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2001 J. Phys. A: Math. Gen. 34 6969

(http://iopscience.iop.org/0305-4470/34/35/322)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 6969–6980 PII: S0305-4470(01)19664-2

Inseparability of mixed two-mode Gaussian states generated with a $SU(1,1)$ interferometer

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Received 27 November 2000
Published 24 August 2001
Online at stacks.iop.org/JPhysA/34/6969

Abstract

We study the entanglement properties of two-mode Gaussian light emerging from a generic $SU(1,1)$ interferometer. Our tool is the two-mode characteristic function which is determined by the 4×4 covariance matrix. For an initial product of two mixed single-mode Gaussian states we investigate the output two-mode covariance matrix. Its structure displays the noise properties of the reduced states as well as the correlations between modes. Classicality of the output two-mode state is characterized by the existence of the Glauber–Sudarshan P representation. For testing separability we apply the Peres–Simon criterion requiring preservation of the positivity of the density matrix under partial transposition. Since inseparability entails nonclassicality, the threshold gain above which nonclassicality of the output state becomes manifest is lower than that allowing for its inseparability. We find that only for a thermal input do nonclassicality and inseparability of the output have the same threshold gain.

PACS numbers: 03.67.-a, 03.65.Ta

1. Introduction

The statistical properties of two-mode light generated in several well-known processes as nondegenerate parametric amplification and degenerate four-wave mixing were intensely studied in order to find evidence for the quantum nature of light. The two-mode radiation resulting from these processes has nonclassical properties such as squeezing and strong correlations between the two beams. A unified treatment of four-port devices having as active elements nondegenerate parametric amplifiers or degenerate four-wave mixers has been initiated by Yurke *et al* [1]. In fact, since such a device performs Bogoliubov transformations of the amplitude operators, they have termed it as a $SU(1,1)$ *interferometer*. In this paper we study the way in which a two-mode $SU(1,1)$-interferometer acts on mixed squeezed-state inputs. This problem meets the recent interest in quantum information processing of Gaussian field states [2,3]. Quantum communication experiments in the near-infrared and optical domain are possible using Gaussian states at the input ports of active devices modelled by a $SU(1,1)$ interferometer [4]. Moreover, it has recently been understood that testing for the preservation of nonlocal entanglement in continuous-variable quantum teleportation [2] involves the need to formulate separability criteria for two-mode Gaussian states [5, 6]. Therefore, we focus on the properties of the two-mode Gaussian output state of the $SU(1,1)$ interferometer when the input is a product of mixed one-mode Gaussian states. We investigate the conditions for classicality (*existence of the P representation*) [7] and separability [8] for the output state. Depending on the input state and the gain of the interferometer, the two-mode output state may be either classical, or nonclassical and still separable, or inseparable. We find gain conditions for reaching all these situations.

The paper is organized as follows. In section 2, we give a brief description of mixed two-mode Gaussian states in terms of their characteristic functions (CFs). The properties of the two-mode covariance matrix studied in our paper are determined by the generalized form of the Robertson–Schrödinger uncertainty relation [5, 9–11]. In section 3 we find the $SU(1,1)$ transformation of the covariance matrix of the two-mode input state. The action of the interferometer imposes a squeezing operation with real parameter $2r$ and phase shiftings for the reduced states. The two-mode output state is studied in section 4. We point out here several properties that arise from the explicit $SU(1,1)$ transformation found in section 3. In section 5 we derive the gain conditions for emergency of nonclassicality in the case of a noisy input. Then we apply the necessary condition for separability proposed by Peres [12]: the matrix obtained by partial transposition of the density matrix should be non-negative. Recently, Simon [5] has proved that Peres' statement is also *a sufficient condition* for separability of two-mode Gaussian states. We find here the gain values above which the initial separable state becomes entangled under the action of the interferometer. Finally, we prove that the violation of the 2-entropy inequality [13, 14] is a sufficient but not a necessary condition for inseparability of the Gaussian states discussed in this paper.

Section 6 summarizes our conclusions.

2. Two-mode Gaussian states

Although the properties of quantum Gaussian states have been largely investigated over the years [5, 9–11, 15–24], we find it useful to point out here a CF description. We deal only with the two-mode case. Note that a general treatment of multimode Gaussian states built on the theory of canonical symplectic forms was given in [5, 9–11, 15, 20, 24].

2.1. Two-mode Gaussian states

We denote by a_1 and a_2 the annihilation operators of the two modes. The two-mode CF is defined as the expectation value of the two-mode Weyl displacement operator

$$\chi(\lambda_1, \lambda_2) = \mathrm{Tr}\,[\rho D_1(\lambda_1) D_2(\lambda_2)] \tag{2.1}$$

with $D(\lambda) = \exp(\lambda a^\dagger - \lambda^* a)$.

The CF of a Gaussian state is

$$\begin{aligned}\chi(\lambda_1, \lambda_2) = {}& \exp\left[-\left(A_1 + \tfrac{1}{2}\right)|\lambda_1|^2 - \tfrac{1}{2}B_1^*\lambda_1^2 - \tfrac{1}{2}B_1(\lambda_1^*)^2 + C_1^*\lambda_1 - C_1\lambda_1^*\right] \\ & \times \exp\left[-\left(A_2 + \tfrac{1}{2}\right)|\lambda_2|^2 - \tfrac{1}{2}B_2^*\lambda_2^2 - \tfrac{1}{2}B_2(\lambda_2^*)^2 + C_2^*\lambda_2 - C_2\lambda_2^*\right] \\ & \times \exp\left[-F\lambda_1^*\lambda_2 - F^*\lambda_1\lambda_2^* + G\lambda_1^*\lambda_2^* + G^*\lambda_1\lambda_2\right].\end{aligned} \tag{2.2}$$

By differentiating the two-mode normally ordered CF

$$\chi^{(N)}(\lambda_1, \lambda_2) := \langle \exp(\lambda_1 a_1^\dagger) \exp(-\lambda_1^* a_1) \exp(\lambda_2 a_2^\dagger) \exp(-\lambda_2^* a_2) \rangle \tag{2.3}$$

with respect to $\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*$ at the point $\lambda_1 = \lambda_1^* = \lambda_2 = \lambda_2^* = 0$, one can obtain the expectation values of interest for the reduced states and their correlations.

Here we deal only with undisplaced Gaussian states, i.e. $C_1 = C_2 = 0$. A meaningful expression of the CF (2.2) is obtained by using the real variables introduced by $\lambda_1 = \frac{1}{\sqrt{2}}(x_2 - \mathrm{i}x_1)$; $\lambda_2 = \frac{1}{\sqrt{2}}(x_4 - \mathrm{i}x_3)$. The CF (2.2) is written compactly as

$$\chi(x) = \exp\{-\tfrac{1}{2}x^T \mathcal{V} x\} \tag{2.4}$$

with x^T the row vector $(x_1\ x_2\ x_3\ x_4)$. x^T denotes the transpose of the column vector x. $\mathcal{V}$ is the real, symmetric and positive 4×4 covariance matrix. It has the following block structure:

$$\mathcal{V} = \left(\begin{array}{c|c} \mathcal{V}_1 & \mathcal{E} \\ \hline \mathcal{E}^T & \mathcal{V}_2 \end{array} \right) \tag{2.5}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{E}$ are 2×2 matrices:

- $\mathcal{V}_1$ and $\mathcal{V}_2$ are covariance matrices for the reduced one-mode states. They contain the variances of the canonical operators ($j = 1, 2$)

$$q_j = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger) \qquad p_j = \frac{1}{\sqrt{2}\mathrm{i}}(a_j - a_j^\dagger). \tag{2.6}$$

From the CF (2.2) we find the structure of the one-mode covariance matrices ($j = 1, 2$):

$$\sigma(q_j, q_j) = A_j + \tfrac{1}{2} - \mathbb{R}(B_j) \tag{2.7a}$$

$$\sigma(p_j, p_j) = A_j + \tfrac{1}{2} + \mathbb{R}(B_j) \tag{2.7b}$$

$$\sigma(q_j, p_j) = \sigma(p_j, q_j) = -\mathbb{I}(B_j). \tag{2.7c}$$

The reductions are squeezed thermal states (STSs) as expected. See [16, 17, 25] for a full account of the properties of one-mode STSs.

- The matrix $\mathcal{E}$ that we term as *the entanglement matrix*, contains the correlations between modes expressed by the variances

$$\sigma(q_1, q_2) := \langle \Delta q_1 \Delta q_2 \rangle = \mathbb{R}(F) + \mathbb{R}(G) \tag{2.8a}$$

$$\sigma(q_1, p_2) := \langle \Delta q_1 \Delta p_2 \rangle = \mathbb{I}(G) - \mathbb{I}(F) \tag{2.8b}$$

$$\sigma(p_1, q_2) := \langle \Delta p_1 \Delta q_2 \rangle = \mathbb{I}(G) + \mathbb{I}(F) \tag{2.8c}$$

$$\sigma(p_1, p_2) := \langle \Delta p_1 \Delta p_2 \rangle = \mathbb{R}(F) - \mathbb{R}(G). \tag{2.8d}$$

A different but equivalent block form of the multimode covariance matrix was adopted in [10,21,24], where the correlations between coordinates are written in a block matrix, another block contains correlations between momenta and the third one contains correlations between coordinates and momenta.

The properties of the two-mode covariance matrix (2.5) are determined by the generalized form of the Robertson–Schrödinger uncertainty relation [9–11, 21, 24]. We follow here the treatment recently exposed in [5, 11] which is in accordance with the way we have written the covariance matrix, equation (2.5): the canonical operators of the two modes are arranged into the four-dimensional row vector $\hat{\xi} = (q_1\ p_1\ q_2\ p_2)$. The commutation relations can be written compactly:

$$[\xi_\alpha, \xi_\beta] = \mathrm{i}\Omega_{\alpha\beta} I \qquad \alpha, \beta = 1, 2, 3, 4. \tag{2.9}$$

The constants $\Omega_{\alpha\beta}$ fill a 4×4 matrix with the structure

$$\Omega = \left(\begin{array}{c|c} \mathcal{J} & 0 \\ \hline 0 & \mathcal{J} \end{array}\right) \tag{2.10}$$

where $\mathcal{J}$ is the 2×2 nondiagonal matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.11}$$

having the property $\mathcal{J}^2 = -I_2$, with I_2 the 2×2 identity matrix.

Now, the Robertson–Schrödinger uncertainty relation in the two-mode case requires the non-negativity of the matrix

$$\mathcal{T} := \mathcal{V} + \frac{\mathrm{i}}{2}\Omega \geqslant 0. \tag{2.12}$$

Equation (2.12) is a restriction that the covariance matrix $\mathcal{V}$ of any state has to satisfy. The condition (2.12) acts on the reduced modes too. We have

$$\det \mathcal{V}_1 \geqslant \tfrac{1}{4} \qquad \det \mathcal{V}_2 \geqslant \tfrac{1}{4} \tag{2.13}$$

in accordance with the uncertainty principle for the one-mode states.

A simple calculation using the Schur formula applied to the matrix (2.5),

$$\det \mathcal{V} = \det \mathcal{V}_1 \det[\mathcal{V}_2 - \mathcal{E}^T \mathcal{V}_1^{-1} \mathcal{E}] \tag{2.14}$$

gives first

$$\det \mathcal{V} = \det \mathcal{V}_1 \det \mathcal{V}_2 + [\det \mathcal{E}]^2 - \mathrm{tr}[\mathcal{V}_1 (\mathrm{adj}\, \mathcal{E})^{\mathrm{T}} \mathcal{V}_2 \,\mathrm{adj}\, \mathcal{E}] \tag{2.15}$$

where $\mathrm{adj}\, \mathcal{E}$ denotes the adjoint matrix of $\mathcal{E}$, i.e. the transpose of its cofactor matrix. Similarly, the physical requirement (2.12) can be written in a form which is invariant under independent local canonical transformations[4]

$$\det \mathcal{V}_1 \det \mathcal{V}_2 + \left(\tfrac{1}{4} - \det \mathcal{E}\right)^2 - \mathrm{tr}[\mathcal{V}_1 (\mathrm{adj}\, \mathcal{E})^{\mathrm{T}} \mathcal{V}_2 \,\mathrm{adj}\, \mathcal{E}] \geqslant \tfrac{1}{4}[\det \mathcal{V}_1 + \det \mathcal{V}_2]. \tag{2.16}$$

The elimination of the local invariant $\mathrm{tr}[\mathcal{V}_1 (\mathrm{adj}\, \mathcal{E})^{\mathrm{T}} \mathcal{V}_2 \,\mathrm{adj}\, \mathcal{E}]$ between equations (2.16) and (2.15) yields a simpler explicit form of the uncertainty principle (2.12)

$$\det \mathcal{V} - \tfrac{1}{4}[\det \mathcal{V}_1 + \det \mathcal{V}_2 + 2 \det \mathcal{E}] + \tfrac{1}{16} \geqslant 0. \tag{2.17}$$

From the degree of purity

$$\mathrm{Tr}\,(\rho^2) = \frac{1}{\pi^2} \int \mathrm{d}^4 x\, |\chi(x)|^2 = \frac{1}{4\sqrt{\det \mathcal{V}}} \leqslant 1 \tag{2.18}$$

we find that the determinant of the covariance matrix has to satisfy

$$\det \mathcal{V} \geqslant \tfrac{1}{16}. \tag{2.19}$$

[4] Simon [5] found this invariant form by local-transforming the covariance matrix to obtain a simpler form in which the reduced 2×2 covariance matrices and the entanglement matrix are diagonal. In [35] this transformed matrix was called the standard form I.

3. The $SU(1,1)$-interferometer

3.1. Solution for a Gaussian input

We will apply the above formalism of the Gaussian CF to the study of the way a two-mode $SU(1,1)$-interferometer acts on squeezed-states inputs. A two-mode $SU(1,1)$-interferometer [1, 26] models several phase-sensitive elements as nondegenerate parametric amplifiers [27–29] and degenerate four-wave mixers [30]. In their paper [1], Yurke *et al* considered the $SU(1,1)$-interferometer as a device with two input ports described by the annihilation operators a_1 and a_2 and two output ports with the annihilation operators denoted by b_1 and b_2. In the Schrödinger picture the output density operator of the two-mode system is [1,26]

$$\rho_{\mathrm{out}} = \mathcal{S}^{\dagger}\rho_{\mathrm{in}}\mathcal{S}. \tag{3.1}$$

Here $\mathcal{S}$ stands for the active action of the four-port device, and, in the most general case, consists of three successive operations:

$$\mathcal{S} = \exp(-\mathrm{i}\phi K_z)\exp(-\mathrm{i}2rK_y)\exp(-\mathrm{i}\psi K_z) \tag{3.2}$$

where:

(a) $\exp(-\mathrm{i}\psi K_z)$ is a rotation of an angle ψ generated by the operator $K_z = \frac{1}{2}(a_1^{\dagger}a_1 + a_2^{\dagger}a_2)$. This unitary transformation produces a common phase shift of the annihilation operators [31,32]

$$\exp(-\mathrm{i}\psi K_z)a_{1,2}\exp(\mathrm{i}\psi K_z) = \exp\left(\mathrm{i}\frac{\psi}{2}\right)a_{1,2}. \tag{3.3}$$

(b) $\exp(-\mathrm{i}2rK_y)$ is a two-mode squeeze operator [31–33] with $K_y = \frac{\mathrm{i}}{2}(a_1a_2 - a_1^{\dagger}a_2^{\dagger})$ and $(2r)$ a real squeeze parameter. The transformed amplitude operators are

$$\exp(-\mathrm{i}2rK_y)a_{1,2}\exp(\mathrm{i}2rK_y) = \cosh r a_{1,2} + \sinh r a_{2,1}^{\dagger}. \tag{3.4}$$

(c) $\exp(-\mathrm{i}\phi K_z)$ is another rotation of angle ϕ. Note that K_y and K_z are two generators of the $SU(1,1)$ representation.

In [26], the transformation of the wavefunction in the position and momentum representations was written using the explicit expression of the active operator (3.2). However, such a treatment is useful only in the case of a pure two-mode state input. Here we use the CF introduced in equation (2.1). We insert equation (3.1) in the definition (2.1) and get successively

$$\begin{aligned}\chi_{\mathrm{out}}(\lambda_1,\lambda_2) &= \mathrm{Tr}[\mathcal{S}^{\dagger}\rho_{\mathrm{in}}\mathcal{S}D_1(\lambda_1)D_2(\lambda_2)]\\ &= \mathrm{Tr}[\rho_{\mathrm{in}}\mathcal{S}D_1(\lambda_1)D_2(\lambda_2)\mathcal{S}^{\dagger}].\end{aligned} \tag{3.5}$$

Now, for the sake of simplicity, we restrict ourselves to the case when the squeezing operation is accompanied by a unique rotation of angle ϕ. Consequently we get

$$\begin{aligned}\chi_{\mathrm{out}}(\lambda_1,\lambda_2) = \chi_{\mathrm{in}}\bigg\{&[\lambda_1\cosh r - \lambda_2^*\sinh r]\exp\left(-\mathrm{i}\frac{\phi}{2}\right),\\ &[-\lambda_1^*\sinh r + \lambda_2\cosh r]\exp\left(-\mathrm{i}\frac{\phi}{2}\right)\bigg\}.\end{aligned} \tag{3.6}$$

As a consequence of equation (3.6), an input two-mode Gaussian state remains Gaussian after $SU(1,1)$ interaction. From now on we consider a Gaussian input product-state,

$$\begin{aligned}\chi_{\mathrm{in}}(\lambda_1,\lambda_2) &= \chi_1(\lambda_1)\chi_2(\lambda_2)\\ &= \exp[-(A_1+\tfrac{1}{2})|\lambda_1|^2 - \tfrac{1}{2}B_1^*\lambda_1^2 - \tfrac{1}{2}B_1(\lambda_1^*)^2]\\ &\quad\times\exp[-(A_2+\tfrac{1}{2})|\lambda_2|^2 - \tfrac{1}{2}B_2^*\lambda_2^2 - \tfrac{1}{2}B_2(\lambda_2^*)^2].\end{aligned} \tag{3.7}$$

We look at equations (3.6) and (3.7) and write down the coefficients of the two-mode output CF which is of the type (2.2):

$$A_1' + \tfrac{1}{2} = (A_1 + \tfrac{1}{2})(\cosh r)^2 + (A_2 + \tfrac{1}{2})(\sinh r)^2 \tag{3.8a}$$

$$A_2' + \tfrac{1}{2} = (A_1 + \tfrac{1}{2})(\sinh r)^2 + (A_2 + \tfrac{1}{2})(\cosh r)^2 \tag{3.8b}$$

$$B_1' = B_1 \exp(\mathrm{i}\phi)(\cosh r)^2 + B_2^* \exp(-\mathrm{i}\phi)(\sinh r)^2 \tag{3.8c}$$

$$B_2' = B_1^* \exp(-\mathrm{i}\phi)(\sinh r)^2 + B_2 \exp(\mathrm{i}\phi)(\cosh r)^2 \tag{3.8d}$$

$$F = -\tfrac{1}{2}[B_1 \exp(\mathrm{i}\phi) + B_2^* \exp(-\mathrm{i}\phi)] \sinh(2r) \tag{3.8e}$$

$$G = \tfrac{1}{2}(A_1 + A_2 + 1)\sinh(2r). \tag{3.8f}$$

As shown in section 3, the covariance matrix can be explicitly written in block form (2.5) by employing equations (2.7) and (2.8). In this case we get the following elements of the basic matrices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{E}$.

- The reduced output mode 1

$$\sigma(q_1, q_1) = \sigma_0(q_1, q_1)(\cosh r)^2 + \sigma_0(q_2, q_2)(\sinh r)^2 \tag{3.9a}$$

$$\sigma(p_1, q_1) = \sigma_0(p_1, q_1)(\cosh r)^2 - \sigma_0(p_2, q_2)(\sinh r)^2 \tag{3.9b}$$

$$\sigma(p_1, p_1) = \sigma_0(p_1, p_1)(\cosh r)^2 + \sigma_0(p_2, p_2)(\sinh r)^2. \tag{3.9c}$$

- The reduced output mode 2. We get similar variances to those given in equations (3.9) by interchanging overall the subscripts 1 and 2.
- The entanglement matrix $\mathcal{E}$

$$\sigma(q_1, q_2) = \tfrac{1}{2}\sinh(2r)[\sigma_0(q_1, q_1) + \sigma_0(q_2, q_2)] \tag{3.10a}$$

$$\sigma(p_1, p_2) = -\tfrac{1}{2}\sinh(2r)[\sigma_0(p_1, p_1) + \sigma_0(p_2, p_2)] \tag{3.10b}$$

$$\sigma(q_1, p_2) = -\tfrac{1}{2}\sinh(2r)[\sigma_0(q_1, p_1) - \sigma_0(q_2, p_2)] \tag{3.10c}$$

$$\sigma(p_1, q_2) = -\sigma(q_1, p_2). \tag{3.10d}$$

In equations (3.9) and (3.10), we have denoted by σ_0 the variances of the input single-mode states with modified phases:

$$\varphi_1' = \varphi_1 + \phi \qquad \varphi_2' = \varphi_2 + \phi. \tag{3.11}$$

For example,

$$\sigma(p_1, p_1) = [A_1 + \tfrac{1}{2} + \mathbb{R}(B_1 \mathrm{e}^{\mathrm{i}\phi})](\cosh r)^2 + [A_2 + \tfrac{1}{2} + \mathbb{R}(B_2^* \mathrm{e}^{-\mathrm{i}\phi})](\sinh r)^2. \tag{3.12}$$

We see that the phases of the output states are controlled by the phase ϕ introduced by the interaction.

4. The two-mode output state. Generalities

We point out that the degree of purity of the input product state, equation (3.7), is left unchanged by the unitary active actions (3.2) of the $SU(1, 1)$ interferometer. Consequently we have

$$\det \mathcal{V} = \det \mathcal{V}^{(0)} = \det \mathcal{V}_1^{(0)} \det \mathcal{V}_2^{(0)} \tag{4.1}$$

which means that the degree of purity of the two-mode state depends only on the thermal noise in the input single-mode states.

Several interesting properties of the matrices $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{E}$ arise from the explicit dependence of their elements, equations (3.9) and (3.10) of the corresponding input ones. We have

$$\det \mathcal{V}_1 = \det \mathcal{V}_1^{(0)}(\cosh r)^2 - \det \mathcal{V}_2^{(0)}(\sinh r)^2 - \det\mathcal{E} \tag{4.2a}$$

$$\det \mathcal{V}_2 = \det \mathcal{V}_2^{(0)}(\cosh r)^2 - \det \mathcal{V}_1^{(0)}(\sinh r)^2 - \det\mathcal{E} \tag{4.2b}$$

where

$$\det \mathcal{E} = -\tfrac{1}{4}[\sinh(2r)]^2 \Sigma_0(p)\Sigma_0(q). \tag{4.2c}$$

In equation (4.2*c*) we have introduced the notations

$$\Sigma_0(q) := \sigma_0(q_1, q_1) + \sigma_0(q_2, q_2) \tag{4.3}$$

$$\Sigma_0(p) := \sigma_0(p_1, p_1) + \sigma_0(p_2, p_2). \tag{4.4}$$

We also obtain

$$\mathrm{tr}\, \mathcal{V}_1 = \mathrm{tr}\, \mathcal{V}_1^{(0)}(\cosh r)^2 + \mathrm{tr}\, \mathcal{V}_2^{(0)}(\sinh r)^2 \tag{4.5a}$$

$$\mathrm{tr}\, \mathcal{V}_2 = \mathrm{tr}\, \mathcal{V}_2^{(0)}(\cosh r)^2 + \mathrm{tr}\, \mathcal{V}_1^{(0)}(\sinh r)^2. \tag{4.5b}$$

In equations (4.2), we have expressed the local invariants for output modes as functions of the similar input ones. The entanglement matrix explicitly occurs in these equations showing that the correlations between modes modify the degree of mixing of the reduced states.

By applying the Robertson–Schrödinger uncertainty relation for the one-mode reduced states, equation (2.13), in the rhs of equations (4.2) we get an important condition that the entanglement matrix has to obey:

$$\det \mathcal{V}_{1,2} + \det \mathcal{E} \geqslant \tfrac{1}{4}. \tag{4.6}$$

We can thus infer that, at least for pure two-mode states (equality in equation (4.6)), we must have $\det \mathcal{E} < 0$.

5. The two-mode output state. Nonclassicality and inseparability

By definition, the density operator characterizing a *separable* state of a bipartite system can be written as a convex combination of product states [8]:

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)} \tag{5.1}$$

where $\rho_i^{(1)}$ and $\rho_i^{(2)}$ are arbitrary local density operators. For a two-mode field state a continuous convex combination is the well known diagonal Glauber–Sudarshan P representation [7]

$$\rho = \int \mathrm{d}^2\gamma_1\, \mathrm{d}^2\gamma_2\, P(\gamma_1, \gamma_2)|\gamma_1\rangle\langle\gamma_1| \otimes |\gamma_2\rangle\langle\gamma_2| \tag{5.2}$$

with $\mathrm{d}^2\gamma = \mathrm{d}(\mathbb{R}\gamma)\mathrm{d}(\mathbb{I}\gamma)$. Quantum states with negative or highly singular P representation provide evidence for the quantum nature of light [34]. In contrast, since its discovery by Glauber and Sudarshan [7], the existence of the P representation as a well behaved function was related to the *classicality* of the quantum state. As $P(\gamma_1, \gamma_2)$ is the Fourier transform of the normally ordered CF, the condition for its existence,

$$\mathcal{V} - \tfrac{1}{2} I_4 \geqslant 0 \tag{5.3}$$

defines the classicality of a quantum two-mode Gaussian state. In equation (5.3) $\mathcal{V}$ is the covariance matrix and I_4 is the 4×4 unity matrix. The semipositiveness condition (5.3) is evidently *sufficient* to ensure separability of the two-mode Gaussian state.

Even if the Glauber–Sudarshan P representation does not exist as a well behaved function the two-mode state could still be separable. For example, a continuous convex combination could be built by using a generalized P representation. We have

$$\rho = \int \mathrm{d}^2\gamma_1\, \mathrm{d}^2\gamma_2\, P'(\gamma_1, \gamma_2) S_1|\gamma_1\rangle\langle\gamma_1|S_1^\dagger \otimes S_2|\gamma_2\rangle\langle\gamma_2|S_2^\dagger \tag{5.4}$$

with $S_{1,2}$ arbitrary squeeze operators. Further, we can write the equivalent equation

$$\rho' := S_1^\dagger S_2^\dagger \rho S_2 S_1 = \int d^2\gamma_1\, d^2\gamma_2\, P'(\gamma_1, \gamma_2)|\gamma_1\rangle\langle\gamma_1| \otimes |\gamma_2\rangle\langle\gamma_2| \tag{5.5}$$

from which we see that $P'(\gamma_1, \gamma_2)$ is the Glauber–Sudarshan P representation for the locally transformed density operator ρ'. The existence of a well behaved $P'(\gamma_1, \gamma_2)$ implies the separability of the states described by ρ and ρ' because they are related by local unitary transformations. However, if they exist, the local squeezing operations achieving such a goal are not easily found. This is one of the reasons for the great interest in formulating criteria for separability of Gaussian states [5, 35]. Another reason is the recent experimental potential of teleporting quantum states of a single-mode electromagnetic field. Experimentally, teleportation of a coherent state has been already reported [36]. Ingenious protocols for teleporting nonclassical states have been recently proposed [2, 6, 37, 38]. In the protocols described by Braunstein and Kimble [2] and Tan [6], the sender and the receiver of a single-mode Gaussian state share an entangled two-mode squeezed state. Therefore, it is an interesting issue to analyse separability properties of such states.

Recently, Simon [5] has examined the Peres–Horodecki separability criterion of preservation of the non-negativity of the density matrix under partial transposition [12,39,40] in the case of bipartite continuous-variable states. Originally, this criterion was proposed as a necessary condition for separability in finite-dimensional Hilbert spaces [12]. For a two-dimensional system, the Peres' statement was proved to be a necessary and sufficient condition for separability [39,40]. Interestingly, Simon found the same result in the infinite-dimensional case of the two-mode Gaussian states[5].

In the formalism of the CF that we have used here, partial transpose of the density matrix with respect to only one subsystem, say mode 2, means

$$\lambda_2 \longrightarrow -\lambda_2^* \tag{5.6}$$

in equation (2.2). It is easy to see that the operation (5.6) modifies only the determinant of the entanglement matrix which changes its sign. Consequently, a separable state has to satisfy the uncertainty principle, equation (2.17), written for $\pm|\det\mathcal{E}|$. In this way, the Peres–Horodecki statement which is a necessary condition for separability has the local-invariant form

$$\det\mathcal{V} - \tfrac{1}{4}[\det\mathcal{V}_1 + \det\mathcal{V}_2 + 2|\det\mathcal{E}|] + \tfrac{1}{16} \geqslant 0. \tag{5.7}$$

To prove that the statement (5.7) is also sufficient for separability Simon has found the explicit local squeezing and rotation operations that afforded the existence of the P representation of the transformed state and thus, the separability of the original one.

In the following we analyse the conditions for nonclassicality (= nonexistence of the P representation) and inseparability for the output of a $SU(1, 1)$ interferometer. We consider squeezed states at the two input ports. However, to have simpler analytical forms we take

$$\varphi_1' = \varphi_1 + \phi = \pi \qquad \varphi_2' = \varphi_2 + \phi = \pi. \tag{5.8}$$

According to equation (5.8) the input states have the equal phases $\varphi_1 = \varphi_2 = \pi - \phi$. Owing to the phase choice (5.8), the input 2×2 covariance matrices are diagonal. For the signal mode we get

$$\sigma_0(q_1, q_1) = (\bar{n}_1 + \tfrac{1}{2})e^{-2r_1} \tag{5.9a}$$

$$\sigma_0(p_1, p_1) = (\bar{n}_1 + \tfrac{1}{2})e^{2r_1} \tag{5.9b}$$

[5] In [5], Simon has employed the Wigner function in order to formulate the Peres criterion in the bipartite Gaussian case.

while the elements of the idler covariance matrix are obtainable from equations (5.9) by changing the index 1 to 2. In the above equations, we have denoted by r_1 and r_2 the squeeze parameters of the noisy input states which have the thermal mean occupancies $\bar{n}_1$ and $\bar{n}_2$, respectively.

According to equations (3.9) and (3.10), the output matrices $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{E}$ are also diagonal.

5.1. P representation

The requirement (5.3) implies that the P representation exists when all the principal minors of the matrix $\mathcal{V} - \frac{1}{2}I_4$ are non-negative,

$$\det(\mathcal{V} - \tfrac{1}{2}I_4)^{(l)} \geqslant 0 \qquad l = 1, 2, 3, 4. \tag{5.10}$$

We have first the conditions which ensure the existence of the P representation of the reduced mode 1:

$$\sigma(q_1, q_1) - \tfrac{1}{2} \geqslant 0 \qquad \sigma(p_1, p_1) - \tfrac{1}{2} \geqslant 0 \tag{5.11}$$

Further, if the reduced mode 1 has a well behaved P representation, then the whole system is in a classical state provided that

$$[\sigma(q_1, q_1) - \tfrac{1}{2}][\sigma(q_2, q_2) - \tfrac{1}{2}] - [\sigma(q_1, q_2)]^2 \geqslant 0 \tag{5.12a}$$

and

$$[\sigma(p_1, p_1) - \tfrac{1}{2}][\sigma(p_2, p_2) - \tfrac{1}{2}] - [\sigma(p_1, p_2)]^2 \geqslant 0. \tag{5.12b}$$

The simultaneous conditions (5.11) and (5.12) via equations (3.9) and (3.10) are equivalent to a double inequality that the gain of the interferometer has to satisfy

$$\frac{\sigma_0(q_2, q_2) + \frac{1}{2}}{\Sigma_0(q)} \leqslant (\cosh r)^2 \leqslant \frac{(\sigma_0(q_2, q_2) + \frac{1}{2})(\sigma_0(q_1, q_1) + \frac{1}{2})}{\Sigma_0(q)}. \tag{5.13}$$

We have two cases here depending on the classicality of the input state.

- If the input mode 1 were squeezed ($\sigma_0(q_1, q_1) < \frac{1}{2}$) the conditions (5.13) cannot be fulfilled irrespective of the state of the input mode 2. Consequently, a squeezed (nonclassical) input of the $SU(1,1)$ interferometer generates a nonclassical two-mode output state. Although the reduced states become classical when a well known gain condition [41] is met, the correlations induced between modes by the $SU(1,1)$ interaction preserve the nonclassical character of the state.
- Recall that the classicality of the input reduced states implies $\sigma_0(q_1, q_1) > \frac{1}{2}$ and $\sigma_0(q_2, q_2) > \frac{1}{2}$. A classical input leads to a classical two-mode output provided that the gain of the interferometer does not exceed the threshold

$$(\cosh r_c)^2 := \frac{(\sigma_0(q_2, q_2) + \frac{1}{2})(\sigma_0(q_1, q_1) + \frac{1}{2})}{\Sigma_0(q)}. \tag{5.14}$$

If the input reduced states were thermal we find from equation (5.14) that the Glauber–Sudarshan P representation ceases existing when

$$(\cosh r)^2 > (\cosh r_{\text{th}})^2 := \frac{(\bar{n}_1 + 1)(\bar{n}_2 + 1)}{\bar{n}_1 + \bar{n}_2 + 1}. \tag{5.15}$$

We conclude that the existence of the output P representation depends on the degree of squeezing of the input state. For sufficiently large values of the gain (above the value (5.14)) the output two-mode state generated by the $SU(1,1)$ interaction is nonclassical.

5.2. Inseparability

We apply here the Peres–Horodecki criterion for separability in the local-invariant form (5.7) derived by Simon. Taking into account equations (4.1) and (4.2) we get the simple condition

$$\left(\det \mathcal{V}_1^{(0)} - \tfrac{1}{4}\right)\left(\det \mathcal{V}_2^{(0)} - \tfrac{1}{4}\right) - |\det \mathcal{E}| \geqslant 0. \tag{5.16}$$

Due to the phase choice (5.8), the determinant of the entanglement matrix is

$$\det \mathcal{E} = -\tfrac{1}{4}[\sinh(2r)]^2 \Sigma_0(p)\Sigma_0(q). \tag{5.17}$$

Therefore we get the necessary and sufficient condition for output inseparability

$$(\sinh(2r))^2 > (\sinh(2r_{\mathrm{s}}))^2 := \frac{4\bar{n}_1\bar{n}_2(\bar{n}_1+1)(\bar{n}_2+1)}{\Sigma_0(p)\Sigma_0(q)} \tag{5.18}$$

which holds whatever the input is squeezed or unsqueezed. We compare the condition assuring inseparability, equation (5.18), to the straightforward one for nonclassicality arising from equation (5.14),

$$(\sinh(2r))^2 > (\sinh(2r_{\mathrm{c}}))^2 = \frac{4\left[(\sigma_0(q_2,q_2))^2 - \frac{1}{4}\right]\left[(\sigma_0(q_1,q_1))^2 - \frac{1}{4}\right]}{[\Sigma_0(q)]^2}. \tag{5.19}$$

The result is

$$r_{\mathrm{c}} \leqslant r_{\mathrm{s}} \leqslant r_{\mathrm{th}}. \tag{5.20}$$

An immediate specialization of equation (5.18) to the case of thermal input gives for the inseparability the same threshold gain (5.15) as for nonclassicality. According to the inequalities (5.20), any amount of squeezing in the input states decreases the value of gain allowing for output inseparability.

5.3. 2*-entropy inequality*

In [13, 14] it is discussed the possibility of defining inseparability in terms of violation of 2-entropy inequality

$$S_2(\rho) \geqslant S_2(\rho_j) \qquad (j = 1, 2). \tag{5.21}$$

In equation (5.21) the 2-entropy

$$S_2(\rho) := -\ln[\mathrm{Tr}(\rho^2)], \tag{5.22}$$

of the whole system has to be greater than that of its two parts. In the two-dimensional case the 2-entropy inequality holds for all separable states and is violated for all pure states [13]. However, in the mixed-state case, the violation of inequality (5.21) was shown to be a sufficient but not a necessary condition for inseparability [13]. Violation of the 2-entropy inequality is written using equation (5.22) as

$$\mathrm{Tr}\,(\rho^2) \geqslant \mathrm{Tr}\,(\rho_j^2) \qquad (j = 1, 2). \tag{5.23}$$

The inequality (5.23) is in fact an expression of the following statement: a two-component state is inseparable if it is purer than any of its reductions [13].

By applying inequality (5.23) we will find the threshold gain of the $SU(1,1)$ interferometer above which the output Gaussian state is inseparable. We take advantage here of the simple relation between the degree of purity of a Gaussian state and the determinant of the covariance matrix. In the two-mode case we use equation (2.18) while for a single-mode state we have $\mathrm{Tr}\,(\rho^2) = [2\sqrt{\det \mathcal{V}}]^{-1}$. The condition (5.23) is simply

$$\det \mathcal{V}_j > 4 \det \mathcal{V} \qquad (j = 1, 2). \tag{5.24}$$

By inserting the above equations (4.1) and (4.2) and using the parametrization (5.8) we get the inseparability condition

$$(\sinh(2r))^2 > (\sinh(2r_e))^2 := \frac{2}{\Sigma_0(q)\Sigma_0(p)}\left[\mathcal{M} + \frac{\mathcal{N}^2}{\Sigma_0(q)\Sigma_0(p)} - |\mathcal{N}|\left(1 + \frac{2\mathcal{M}}{\Sigma_0(q)\Sigma_0(p)} + \frac{\mathcal{N}^2}{(\Sigma_0(q)\Sigma_0(p))^2}\right)^{1/2}\right] \tag{5.25}$$

where

$$\mathcal{M} := 8(\bar{n}_1 + \tfrac{1}{2})^2(\bar{n}_2 + \tfrac{1}{2})^2 - (\bar{n}_1 + \tfrac{1}{2})^2 - (\bar{n}_2 + \tfrac{1}{2})^2 \tag{5.26}$$

$$\mathcal{N} := (\bar{n}_1 + \tfrac{1}{2})^2 - (\bar{n}_2 + \tfrac{1}{2})^2. \tag{5.27}$$

A comparison between the inseparability condition arising from the violation of the 2-entropy inequality, equation (5.25), and that obtained using the Peres–Simon criterion, equation (5.18), yields the result

$$r_s \leqslant r_e. \tag{5.28}$$

Therefore, in the case of Gaussian states discussed here, the Horodecki's statement proves to be more restrictive than Peres criterion, being *a sufficient* condition for inseparability. Equation (5.28) becomes an equality if and only if one of the reduced input states is pure. If, for example, $\bar{n}_1 = 0$ we get $r_c = r_s = r_e = 0$.

6. Conclusions

In this paper we have examined the problem of inseparability of mixed two-mode Gaussian states obtained as output of a $SU(1,1)$ interferometer. We have found the following features:

- A squeezed (nonclassical) input generates a nonclassical two-mode output state for any gain.
- A classical input at one of the ports leads to a nonclassical two-mode output provided that the gain of the interferometer exceeds the threshold (5.14), which depends on the initial squeezing properties.
- The onset gain for inseparability is greater than that for nonclassicality. However, in the important case of a thermal input nonclassicality and inseparability have the same range of gain.
- Violation of the 2-entropy inequality proves to be a sufficient but not a necessary condition for inseparability.
- Above the threshold (5.18), the $SU(1,1)$ interferometer is a device that produces nonseparable (entangled) two-mode mixed states.

Acknowledgment

We gratefully acknowledge partial support for this work in the form of the research grant no 33088/202 from Romanian CNCSIS.

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